Some Applications of Hawkes Processes for Order Book Modelling

Ioane Muni Toke

Ecole Centrale Paris
Chair of Quantitative Finance

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Introduction : Modeling with Point processes

Self- and mutually exciting Hawkes processes

A simple model for buy and sell intensities

Some statistical findings about the order flows

An order book model with Hawkes processes
Introduction : Modeling with Point processes

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Modeling Financial Data with Point Processes

- (Ultra-) High frequency data i.e. tick-by-tick data
- (Engle 2000) often cited as seminal paper (1996)
- ACD models by (Engle & Russell 1997)
- Empirical fits on durations: Weibull, ...
- Tractability in multivariate settings

Figure: Illustration of a simple point process: events, durations, counter
Intensity process

Let $N$ be a point process adapted to a filtration $\mathcal{F}_t$. The left-continuous $\mathcal{F}_t$-intensity process is defined as

$$\lambda(t|\mathcal{F}_t) = \lim_{h \downarrow 0} \mathbb{E} \left[ \frac{N(t+h) - N(t)}{h} \middle| \mathcal{F}_t \right],$$

or equivalently

$$\lambda(t|\mathcal{F}_t) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P} \left[ \frac{N(t+h) - N(t)}{h} > 0 \middle| \mathcal{F}_t \right].$$
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Multidimensional Hawkes processes

Let $M \in \mathbb{N}^*$. Let $\{(t^m_i)\}_{m=1,...,M}$ be a $M$-dimensional point process. $N_t = (N_t^1, \ldots, N_t^M)$ denotes the associated counting process.

**Definition**

A multidimensional Hawkes process is defined with intensities $\lambda^m, m = 1, \ldots, M$ given by:

$$\lambda^m(t) = \lambda^m_0(t) + \sum_{n=1}^{M} \int_0^t \sum_{j=1}^{P} \alpha_{mn} e^{-\beta_{mn}(t-s)} dN^m_s,$$

i.e. in its simplest version with $P = 1$ and $\lambda^m_0(t)$ constant:

$$\lambda^m(t) = \lambda^m_0 + \sum_{n=1}^{M} \int_0^t \alpha_{mn} e^{-\beta_{mn}(t-s)} dN^m_s = \lambda^m_0 + \sum_{n=1}^{M} \sum_{t^n_i < t} \alpha_{mn} e^{-\beta_{mn}(t-t^n_i)}.$$
Stationarity condition (I)

We’ll take here $P = 1$ to simplify the notations. Rewriting equation (4) using vectorial notation, we have:

$$\lambda(t) = \lambda_0 + \int_{-\infty}^{t} G(t - s) dN_s,$$

(5)

where

$$G(t) = \left( \alpha^{mn} e^{-\beta^{mn} t} 1_{\mathbb{R}+} (t) \right)_{m,n=1,...,M}.$$

(6)

Assuming stationarity gives $E[\lambda(t)] = \mu$ constant vector, and thus stationary intensities must satisfy:

$$\mu = \lambda_0 + E \left[ \int_{-\infty}^{t} G(t - s) dN_s \right] = \lambda_0 + E \left[ \int_{-\infty}^{t} G(t - s) \lambda(s) ds \right],$$

(7)

i.e.

$$\mu = \left( I - \int_{0}^{\infty} G(u) du \right)^{-1} \lambda_0$$

(8)
Stationarity condition (II)

Stationarity of a multivariate Hawkes process

A sufficient condition for a multivariate Hawkes process to be linear is that the spectral radius of the matrix

\[
\Gamma = \int_0^\infty G(u) du = \left( \frac{\alpha_{mn}}{\beta_{mn}} \right)_{m,n=1,\ldots,M} \tag{9}
\]

be strictly smaller than 1.

We recall that the spectral radius of the matrix \( G \) is defined as:

\[
\rho(G) = \max_{a \in S(G)} |a|, \tag{10}
\]

where \( S(G) \) denotes the set of all eigenvalues of \( G \).

Note that this result can also be seen as a particular (linear) case of (Brémaud 1996, Theorem 7) which deals with general non-linear Hawkes processes.
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Thinning procedure

Lewis & Shedler (1979) proposes a “thinning procedure” that allows the simulation of a point process with bounded intensity.

Basic thinning theorem

Consider a one-dimensional non-homogeneous Poisson process \( \{N^*(t)\}_{t \geq 0} \) with rate function \( \lambda^*(t) \), so that the number of points \( N^*(T_0) \) in a fixed interval \((0, T_0]\) has a Poisson distribution with parameter \( \mu^*_0 = \int_0^{T_0} \lambda^*(s)ds \). Let \( t_1^*, t_2^*, \ldots, t_{N^*(T_0)}^* \) be the points of the process in the interval \((0, T_0]\). Suppose that for \( 0 \leq t \leq T_0 \), \( \lambda(t) \leq \lambda^*(t) \).

For \( i = 1, 2, \ldots, N^*(T_0) \), delete the points \( t_i^* \) with probability \( 1 - \frac{\lambda(t_i^*)}{\lambda^*(t_i^*)} \).

Then the remaining points form a non-homogeneous Poisson process \( \{N(t)\}_{t \geq 0} \) with rate function \( \lambda(t) \) in the interval \((0, T_0]\).
Simulation of a multivariate Hawkes process (I)

A general algorithm based on thinning is proposed by Ogata (1981). We use the following notation:

- $\mathcal{U}_{[0,1]}$ denotes the uniform distribution on the interval $[0, 1]$,
- $[0, T]$ is the time interval on which the process is to be simulated,

and we define

$$I^K(t) = \sum_{n=1}^{K} \lambda^n(t)$$

the sum of the intensities of the first $K$ components of the multivariate process. $I^M(t) = \sum_{n=1}^{M} \lambda^n(t)$ is thus the total intensity of the multivariate process and we set $I^0 = 0$. The algorithm is then rewritten as follows.
Simulation of a multivariate Hawkes process (II)

Algorithm - Initialization

1. **Initialization**: Set \( i \leftarrow 1, \ i^1 \leftarrow 1, \ldots, \ i^M \leftarrow 1 \) and
   
   \[
   I^* \leftarrow I^M(0) = \sum_{n=i}^{M} \lambda_n^i(0).
   \]

2. **First event**: Generate \( U \sim \mathcal{U}[0,1] \) and set \( s \leftarrow -\frac{1}{\lambda^*} \ln U \).
   
   1. If \( s > T \) Then go to last step.
   2. **Attribution Test**: Generate \( D \sim \mathcal{U}[0,1] \) and set \( t_{n_0}^{i_1} \leftarrow s \) where \( n_0 \) is such that
      
      \[
      I^{n_0-1}(0) < D \leq I^{n_0}(0).
      \]
   3. Set \( t_1 \leftarrow t_{1}^{n_0} \).
Algorithm - General routine

3 General routine: Set $i^{n_0} \leftarrow i^{n_0} + 1$ and $i \leftarrow i + 1$.

1 Update maximum intensity: Set $I^* \leftarrow I^M(t_{i-1}) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{nn_0}$.

2 New event: Generate $U \sim \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{I^*} \ln U$.
   If $s > T$, Then go to the last step.

3 Attribution-Rejection test: Generate $D \sim \mathcal{U}_{[0,1]}$.
   If $D \leq \frac{I^M(s)}{I^*}$,
   Then set $t_{i^{n_0}}^{n_0} \leftarrow s$ where $n_0$ is such that $\frac{I^{n_0-1}(s)}{I^*} < D \leq \frac{I^{n_0}(s)}{I^*}$, and $t_i \leftarrow t_{i^{n_0}}^{n_0}$ and go through the general routine again,
   Else update $I^* \leftarrow I^M(s)$ and try a new date at step (b) of the general routine.

4 Output: Retrieve the simulated process $\{t_i^n\}_{i=1,...,M}$ on $[0, T]$. 
Sample paths of a bivariate Hawkes process (I)

We simulate a bivariate Hawkes process with $P = 1$ and the following parameters:

\[
\begin{align*}
\lambda_0^1 &= 0.1, \quad \alpha_{11}^1 = 0.2, \quad \beta_{11}^1 = 1.0, \quad \alpha_{12}^1 = 0.1, \quad \beta_{12}^1 = 1.0, \\
\lambda_0^2 &= 0.5, \quad \alpha_{21}^1 = 0.5, \quad \beta_{21}^1 = 1.0, \quad \alpha_{22}^1 = 0.1, \quad \beta_{22}^1 = 1.0,
\end{align*}
\]  

(12)
Sample paths of a bivariate Hawkes process (II)

Figure: Simulation of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (12).
Sample paths of a bivariate Hawkes process (III)

Figure: Simulation of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (12). (Zoom of the previous figure).
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Computation of the log-likelihood function (I)

The log-likelihood of a multidimensional Hawkes process can be computed as the sum of the likelihood of each coordinate, i.e. is written:

$$\ln \mathcal{L}(\{t_i\}_{i=1,...,N}) = \sum_{m=1}^{M} \ln \mathcal{L}^m(\{t_i\}),$$  \hspace{1cm} (13)

where each term is defined by:

$$\ln \mathcal{L}^m(\{t_i\}) = \int_0^T (1 - \lambda^m(s)) \, ds + \int_0^T \ln \lambda^m(s) \, dN^m(s).$$  \hspace{1cm} (14)
In the case of a multidimensional Hawkes process, denoting \( \{ t_i \}_{i=1,...,N} \) the ordered pool of all events \( \{ \{ t_i^m \}_{m=1,...,M} \} \), this log-likelihood can be computed as:

\[
\ln \mathcal{L}^m(\{ t_i \}) = T - \Lambda^m(0, T)
+ \sum_{i=1}^{N} z_i^m \ln \left[ \lambda_0^m(t_i) + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < t_i} \alpha_{jn}^m e^{-\beta_{jn}^m(t_i-t_k^n)} \right],
\]

where \( z_i^m \) is equal to 1 if the event \( t_i \) is of type \( m \), 0 otherwise.
Computation of the log-likelihood function (III)

This can be computed in a recursive way. We observe that:

\[ R_{jm}^{mn}(l) = \sum_{t_k^n < t_l^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} \]

\[ = \sum_{t_k^n < t_{l-1}^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} \]

\[ = e^{-\beta_{jm}^{mn}(t_{l-1}^m - t_{l-1}^m)} \sum_{t_k^n < t_{l-1}^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} \]

\[ = e^{-\beta_{jm}^{mn}(t_{l-1}^m - t_{l-1}^m)} R_{jm}^{mn}(l - 1) + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} \]

\[ = \begin{cases} 
    e^{-\beta_{jm}^{mn}(t_{l-1}^m - t_{l-1}^m)} R_{jm}^{mn}(l - 1) + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_{jm}^{mn}(t_l^m - t_k^n)} & \text{if } m \neq n, \\
    e^{-\beta_{jm}^{mn}(t_{l-1}^m - t_{l-1}^m)} \left(1 + R_{jm}^{mn}(l - 1)\right) & \text{if } m = n.
\end{cases} \]
Computation of the log-likelihood function (IV)

The final expression of the log-likelihood may be written:

\[
\ln L^m(\{t_i\}) = T - \sum_{i=1}^{N} \sum_{n=1}^{M} \sum_{j=1}^{P} \frac{\alpha_j^{mn}}{\beta_j^{mn}} \left( 1 - e^{-\beta_j^{mn}(T-t_i)} \right) \\
+ \sum_{t_i^m} \ln \left[ \lambda_0^m(t_i^m) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{mn} R_j^{mn}(l) \right],
\]

where \( R_j^{mn}(l) \) is defined with equation (16) and \( R_j^{mn}(0) = 0 \).
Properties of the maximum-likelihood estimator

Ogata (1978) shows that for a stationary one-dimensional Hawkes process with constant $\lambda_0$ and $P = 1$, the maximum-likelihood estimator $\hat{\theta}^T = (\hat{\lambda}_0, \hat{\alpha}_1, \hat{\beta}_1)$ is

- **consistent**, i.e. converges in probability to the true values $\theta = (\lambda_0, \alpha_1, \beta_1)$ as $T \to \infty$:

  $$\forall \epsilon > 0, \lim_{T \to \infty} P[|\hat{\theta}^T - \theta| > \epsilon] = 0.$$  

(18)

- **asymptotically normal**, i.e.

  $$\sqrt{T} \left( \hat{\theta}^T - \theta \right) \to \mathcal{N}(0, I^{-1}(\theta)).$$  

(19)

where $I^{-1}(\theta) = \left( E \left[ \frac{1}{\lambda} \frac{\partial \lambda}{\partial \theta_i} \frac{\partial \lambda}{\partial \theta_j} \right] \right)_{i,j}$.

- **asymptotically efficient**, i.e. asymptotically reaches the lower bound of the variance.
# Numerical estimation of a simulated process

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<th>$T$</th>
<th>$\lambda_0^1$</th>
<th>$\alpha_{11}^1$</th>
<th>$\beta_{11}^1$</th>
<th>$\alpha_{12}^1$</th>
<th>$\beta_{12}^1$</th>
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<th>$\alpha_{21}^2$</th>
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<th>$\alpha_{22}^2$</th>
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<td>0.369</td>
<td>1.482</td>
<td>1.710</td>
<td>0.518</td>
<td>0.337</td>
<td>0.600</td>
<td>1.605</td>
<td>2.595</td>
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<tr>
<td></td>
<td>(0.372)</td>
<td>(0.268)</td>
<td>(0.269)</td>
<td>(1.216)</td>
<td>(3.172)</td>
<td>(0.272)</td>
<td>(0.206)</td>
<td>(0.365)</td>
<td>(2.051)</td>
<td>(6.586)</td>
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<tr>
<td>500</td>
<td>0.516</td>
<td>0.505</td>
<td>0.268</td>
<td>1.043</td>
<td>0.865</td>
<td>0.518</td>
<td>0.264</td>
<td>0.507</td>
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<td>1.048</td>
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<tr>
<td></td>
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<td>(0.085)</td>
<td>(0.080)</td>
<td>(0.214)</td>
<td>(0.479)</td>
<td>(0.120)</td>
<td>(0.080)</td>
<td>(0.084)</td>
<td>(0.278)</td>
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<td>0.492</td>
<td>0.254</td>
<td>1.018</td>
<td>0.761</td>
<td>0.513</td>
<td>0.255</td>
<td>0.488</td>
<td>0.794</td>
<td>1.003</td>
</tr>
<tr>
<td></td>
<td>(0.079)</td>
<td>(0.054)</td>
<td>(0.052)</td>
<td>(0.122)</td>
<td>(0.203)</td>
<td>(0.092)</td>
<td>(0.052)</td>
<td>(0.061)</td>
<td>(0.387)</td>
<td>(0.152)</td>
</tr>
</tbody>
</table>

| 1000 | 0.500         | 0.500          | 0.250         | 1.000          | 0.750         | 0.500         | 0.250          | 0.500         | 0.750          | 1.000         |

**Table:** Maximum likelihood estimation of a two-dimensional Hawkes process on simulated data. Each estimation is the average result computed on 100 samples of length $[0, T]$. Standard deviations are given in parentheses.
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Time change theorem and goodness of fit

Time change theorem

Let \((N_t)\) be a one-dimensional point process on \(\mathbb{R}_+\) such that 
\[
\int_0^\infty \lambda(s) ds = \infty.
\]
Let \(t_\tau\) be the stopping time defined by
\[
\int_0^{t_\tau} \lambda(s) ds = \tau. \tag{20}
\]
Then the process \(\tilde{N}(\tau) = N(t_\tau)\) is an homogeneous Poisson process with constant intensity \(\lambda = 1\).

(Multivariate componentwise) Corollary

The durations \(\tau_i^m - \tau_{i-1}^m = \Lambda^m(t_{i-1}^m, t_i^m) = \int_{t_{i-1}^m}^{t_i^m} \lambda^m(s) ds\) are exponentially distributed with parameter 1. See e.g. (Bowsher 2007).
Testing the simulated data (I)

The integrated intensity of the $m$-th coordinate of a multidimensional Hawkes process between two consecutive events $t_{i-1}^m$ and $t_i^m$ of type $m$ is computed as:

$$\Lambda^m(t_{i-1}^m, t_i^m) = \int_{t_{i-1}^m}^{t_i^m} \lambda^m(s) ds$$

$$= \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s) ds + \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^{n}<s} \alpha_{jm}^{mn} e^{-\beta_{jm}^{mn}(s-t_k^{n})} ds$$

$$= \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s) ds + \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^{n}<t_i^m} \alpha_{jm}^{mn} e^{-\beta_{jm}^{mn}(s-t_k^{n})} ds$$

$$+ \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_{i-1}^m \leq t_k^{n} < s} \alpha_{jm}^{mn} e^{-\beta_{jm}^{mn}(s-t_k^{n})} ds$$
Testing the simulated data (II)

\[ \Lambda^m(t_{i-1}^m, t_i^m) = \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s)ds \]

\[ + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < t_i^m} \frac{\alpha_{jn}^{mn}}{\beta_j^{mn}} \left[ e^{-\beta_j^{mn}(t_i^m - t_k^n)} - e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)} \right] \]

\[ + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_{i-1}^m \leq t_k^n < t_i^m} \frac{\alpha_{jn}^{mn}}{\beta_j^{mn}} \left[ 1 - e^{-\beta_j^{mn}(t_i^m - t_k^n)} \right]. \] (21)

This computation can be simplified with a recursive element. Let us denote

\[ A_{jn}^{mn}(i - 1) = \sum_{t_k^n < t_i^m - 1} e^{-\beta_j^{mn}(t_i^m - t_k^n)}. \] (22)
Testing the simulated data (III)

We observe that

\[ A_{j}^{mn}(i - 1) = \sum_{t_{k}^{n} < t_{i-1}^{m}} e^{-\beta_{j}^{mn}(t_{i-1}^{m} - t_{k}^{n})} \]

\[ = e^{-\beta_{j}^{mn}(t_{i-1}^{m} - t_{i-2}^{m})} \sum_{t_{k}^{n} < t_{i-2}^{m}} e^{-\beta_{j}^{mn}(t_{i-2}^{m} - t_{k}^{n})} \]

\[ + \sum_{t_{i-2}^{m} \leq t_{k}^{n} < t_{i-1}^{m}} e^{-\beta_{j}^{mn}(t_{i-1}^{m} - t_{k}^{n})} \]

\[ = e^{-\beta_{j}^{mn}(t_{i-1}^{m} - t_{i-2}^{m})} A_{j}^{mn}(i - 2) \]

\[ + \sum_{t_{i-2}^{m} \leq t_{k}^{n} < t_{i-1}^{m}} e^{-\beta_{j}^{mn}(t_{i-1}^{m} - t_{k}^{n})}. \]  (23)
Testing the simulated data (IV)

The integrated density can thus be written $\forall i \in \mathbb{N}^*$:

$$\Lambda^m(t^m_{i-1}, t^m_i) = \int_{t^m_{i-1}}^{t^m_i} \lambda^m_0(s)ds + \sum_{n=1}^{M} \sum_{j=1}^{P} \frac{\alpha_{mn}^j}{\beta_{mn}^j} \left[ \left(1 - e^{-\beta_{mn}^j(t^m_i - t^m_{i-1})}\right) \right. \\
\times A_{jmn}(i - 1) + \sum_{t^m_{i-1} \leq t^n_k < t^m_i} \left(1 - e^{-\beta_{jmn}(t^m_i - t^n_k)}\right), \quad (24)$$

where $A$ is defined as in equation (22) with $\forall j, A_{jmn}(0) = 0.$
Figure: Quantile plots for one sample of simulated data of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (12). (Left) $m = 0$. (Right) $m = 1$. 
# A simple model for buy and sell intensities

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5. An order book model with Hawkes processes
Hewlett (2006) proposes to model the clustered arrivals of buy and sell trades using Hawkes processes. Using the exponent \( B \) for buy variables and \( S \) for sell variables, the model is written:

\[
\lambda^B(t) = \lambda^B_0 + \int_0^t \alpha^{BB} e^{-\beta^{BB}(t-u)} dN^B_u + \int_0^t \alpha^{BS} e^{-\beta^{BS}(t-u)} dN^S_u, \tag{25}
\]

\[
\lambda^S(t) = \lambda^S_0 + \int_0^t \alpha^{SB} e^{-\beta^{SB}(t-u)} dN^B_u + \int_0^t \alpha^{SS} e^{-\beta^{SS}(t-u)} dN^S_u. \tag{26}
\]
Hewlett (2006) imposes some symmetry constraints, stating that mutual excitation and self-excitation should be the same for both processes, which is written:

\[
\begin{align*}
\lambda_0^B &= \lambda_0^S &= \lambda_0 \\
\alpha^{SB} &= \alpha^{BS} &= \alpha^{\text{cross}} \\
\beta^{SB} &= \beta^{BS} &= \beta^{\text{cross}} \\
\alpha^{SS} &= \alpha^{BB} &= \alpha^{\text{self}} \\
\beta^{SS} &= \beta^{BB} &= \beta^{\text{self}}
\end{align*}
\]
Goodness of fit

Hewlett (2006) fits this model on two-month data of EUR/PLN transactions (no dates given): the Hawkes model is a much better fit of the empirical data than the Poisson model.

Figure: Quantile plots of integrated intensities for the Hawkes model (left) and a Poisson model (right) on EUR/PLN buy and sell data. Reproduced from (Hewlett 2006).
Numerical results

The numerical values obtained are:

\[ \lambda_0 = 0.0033, \alpha^{cross} = 0, \alpha^{self} = 0.0169, \beta^{self} = 0.0286. \]  \hspace{1cm} (32)

In other words,

- the occurrence of a buy (resp. sell) order has an exciting effect on the stream of buy (resp. sell) orders, with a typical half-life of \( \frac{\ln 2}{\beta^{self}} \approx 24 \) seconds;
- the zero value of \( \alpha^{cross} \) tends to indicate that there is no influence of buy orders on sell orders, and conversely.
We perform the fit of a bivariate Hawkes model on buy/sell market orders on the following data: BNPP.PA, Feb. 1st 2010 to Feb. 23rd, 2010 (14 trading days), 10am-12am without symmetry constraints. Numerical results are:

\[
\begin{align*}
\lambda_0^B &= 0.080, \quad \alpha^{BB} = 3.230, \quad \beta^{BB} = 13.304, \quad \alpha^{BS} = 0.276, \quad \beta^{BS} = 6.193 \\
\lambda_0^B &= 0.086, \quad \alpha^{SB} = 0.515, \quad \beta^{SB} = 13.451, \quad \alpha^{SS} = 3.789, \quad \beta^{SS} = 14.151
\end{align*}
\]

- Confirmation of the very limited cross-excitation effect.
- Change of magnitude of parameters \( \beta \): difference in precision of data (second, millisecond)
A simple model for buy and sell intensities

Test on our own data (II)

Figure: Quantile plots of integrated intensities for a bivariate Hawkes model on buy/sell market orders fitted on 13 trading days of the stock BNPP.PA (from Feb. 1st 2010 to Feb. 22nd, 2010), 10am-12am each day.
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Classifying orders according to their aggressiveness

Following classical typologies used in microstructure, events occurring in an order book are classified in ten categories:

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
<th>Aggressiveness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Market order that moves the ask</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>Market order that moves the bid</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>Limit order that moves the ask</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>Limit order that moves the bid</td>
<td>Yes</td>
</tr>
<tr>
<td>5</td>
<td>Market order that doesn’t move the ask</td>
<td>No</td>
</tr>
<tr>
<td>6</td>
<td>Market order that doesn’t move the bid</td>
<td>No</td>
</tr>
<tr>
<td>7</td>
<td>Limit order that doesn’t move the ask</td>
<td>No</td>
</tr>
<tr>
<td>8</td>
<td>Limit order that doesn’t move the bid</td>
<td>No</td>
</tr>
<tr>
<td>9</td>
<td>Cancellation at ask</td>
<td>No</td>
</tr>
<tr>
<td>10</td>
<td>Cancellation at bid</td>
<td>No</td>
</tr>
</tbody>
</table>
A 10-variate Hawkes model for aggressive orders

Events of type 1 to 4 are Hawkes processes whose intensities depend on the 10 different sorts of events, i.e. can be written for $m = 1, \ldots, 4$:

$$\lambda^m(t) = \lambda_0(t) + \sum_{n=1}^{10} \int_0^t \alpha_{mn} e^{-\beta_{mn}(t-u)} dN^n_u. \quad (33)$$
Hawkes parameters for aggressive limit orders

**Figure:** Representation of the influences on aggressive limit orders measured by the fitting of a Hawkes model on the Barclay’s order book on January 2002. $\beta^{mn}$ are in abscissas, $\alpha^{mn}$ are in ordinates, the size of the discs are proportional to the number of observed events. Reproduced from (Large 2007).
Hawkes parameters for aggressive market orders

Figure: Representation of the influences on aggressive market orders measured by the fitting of a Hawkes model on the Barclay’s order book on January 2002. $\beta^{mn}$ are in abscissas, $\alpha^{mn}$ are in ordinates, the size of the discs are proportional to the number of observed events. Reproduced from (Large 2007).
Some statistical findings about the order flows

Aggressive market orders on 2010 CAC40 data

Hawkes Parameters For Aggressive Market Orders

**Figure:** Hawkes parameters for aggressive market orders for various CAC40 stocks. These values are computed using median-MLE estimation on 14 days of trading (Feb.1st-Feb.23rd 2010), 10am-12pm.
Some statistical findings about the order flows

Aggressive limit orders on 2010 CAC40 data

Hawkes Parameters For Aggressive Limit Orders

Figure: Hawkes parameters for aggressive limit orders for various CAC40 stocks. These values are computed using median-MLE estimation on 14 days of trading (Feb.1st-Feb.23rd 2010), 10am-12pm.
Passive market orders on 2010 CAC40 data

Figure: Hawkes parameters for passive market orders for various CAC40 stocks. These values are computed using median-MLE estimation on 14 days of trading (Feb.1st-Feb.23rd 2010), 10am-12pm.
Empirical conclusions

Previous figures can be used to draw some conclusions on the way order book events influence each other. The main findings are the followings:

- passive limit orders can be seen as “background noise”;
- aggressive LO are firstly influenced by aggressive MO (“resiliency”);
- aggressive LO are secondly influenced by passive MO;
- aggressive LO are thirdly influenced by aggressive LO;
- aggressive MO are firstly influenced by passive MO;
- aggressive MO are secondly influenced by aggressive MO;
- aggressive MO are thirdly influenced by aggressive LO (“rush to liquidity”);
- passive MO are influenced by passive and aggressive market orders, but not by LO.
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The basic Zero-Intelligence Poisson model ("HP")

**Liquidity provider**

1. arrival of new limit orders: homogeneous Poisson process $N^L(\lambda^L)$
2. arrival of cancellation of orders: homogeneous Poisson process $N^C(\lambda^C)$
3. new limit orders' placement: Student’s distribution with parameters $(\nu^P_1, m^P_1, s^P_1)$ around the same side best quote
4. volume of new limit orders: exponential distribution $\mathcal{E}(1/m^V_1)$
5. in case of a cancellation, orders are deleted with probability $\delta$

**Noise trader (liquidity taker)**

1. arrival of market orders: homogeneous Poisson process $N^M(\mu)$
2. volume of market orders: exponential distribution with mean $\mathcal{E}(1/m^V_2)$. 
Need for physical time in order book models

**Figure:** Empirical density function of the distribution of the bid-ask spread in event time and in physical time.
Measures of inter arrival times

- **(red)** Inter arrival times of the counting process of all orders (limit orders and market orders mixed), i.e. the time step between any order book event (other than cancellation)
- **(green)** Interval time between a market order and immediately following limit order
Empirical evidence of “market making”

**Figure:** Empirical density function of the distribution of the time steps between two consecutive orders (any type, market or limit) and empirical density function of the distribution of the time steps between a market order and the immediately following limit order. X-axis is scaled in seconds. In insets, same data using a semi-log scale. Studied assets: BNPP.PA (left). Reproduced from (Muni Toke 2011).
### Adding dependance between order flows (I)

#### Liquidity provider

1. Arrival of new limit orders: Hawkes process $N^L(\lambda^L)$
2. Arrival of cancellation of orders: homogeneous Poisson process $N^C(\lambda^C)$
3. New limit orders' placement: Student’s distribution with parameters $(\nu_1^P, m_1^P, s_1^P)$ around the same side best quote
4. Volume of new limit orders: exponential distribution $\mathcal{E}(1/m_1^V)$
5. In case of a cancellation, orders are deleted with probability $\delta$

#### Noise trader (liquidity taker)

1. Arrival of market orders: Hawkes process $N^M(\mu)$
2. Volume of market orders: exponential distribution with mean $\mathcal{E}(1/m_2^V)$. 

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Ioane Muni Toke (ECP)  
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Adding dependence between order flows (II)

Hawkes processes $N^L$ and $N^M$

\[
\begin{align*}
\mu(t) &= \mu_0 + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN^M_s \\
\lambda^L(t) &= \lambda_0^L + \int_0^t \alpha_{LM} e^{-\beta_{LM}(t-s)} dN^M_s + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN^L_s 
\end{align*}
\]

- MM and LL effect for clustering of orders
- LM effect as observed on data
- no ML effect
Impact on arrival times (I)

**Figure:** Empirical density function of the distribution of the interarrival times of market orders (left) and limit orders (right) for three simulations, namely HP, MM, LL, compared to empirical measures. In inset, same data using a semi-log scale. Reproduced from (Muni Toke 2011).
Impact on arrival times (II)

**Figure**: Empirical density function of the distribution of the interval times between a market order and the following limit order for three simulations, namely HP, MM+LL, MM+LL+LM, compared to empirical measures. In inset, same data using a semi-log scale. Reproduced from (Muni Toke 2011).
Impact on the bid-ask spread (I)

**Figure:** Empirical density function of the distribution of the bid-ask spread for three simulations, namely HP, MM, MM+LM, compared to empirical measures. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro). Reproduced from (Muni Toke 2011).
Impact on the bid-ask spread (II)

Figure: Empirical density function of the distribution of bid-ask spread for three simulations, namely HP, MM+LL, MM+LL+LM. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro). Reproduced from (Muni Toke 2011).
Conclusion

- Self- and mutually exciting processes (epidemic, earthquakes, ... finance)
- Exponential kernel allows easy manipulation (simulation, estimation)
- Quite good fit on tested data (buy/sell, market/limit)
- See (Bowsher 2007) for a generalized econometric framework
- Lots of possible models/strategies to be imagined
- See (Bacry, Delattre, Hoffmann & Muzy 2011) for a microstructure toy model addressing stylized facts


