Interest rates stochastic models

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Course outline

Useful bibliography

This short course uses material from :

- Brigo D. and Mercurio F. (2006). *Interest rates models - Theory and Practice*, 2nd edition, Springer.
- Martellini L. and Priaulet P. (2000). *Produits de taux d'intérêt*, Economica.
- Shreve S. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer.
- Original research papers (references below).

Part I

Basic concepts

Spot interest rates

- $r(t)$: Instantaneous (interbank) rate, or short rate.
- $P(t,T)$: Price at time t of a T-maturity zero-coupon bond
- $R(t,T)$: Continuously-compounded spot interest rate

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} \text{ i.e. } P(t,T) = e^{-R(t,T)(T-t)} \tag{1}$$

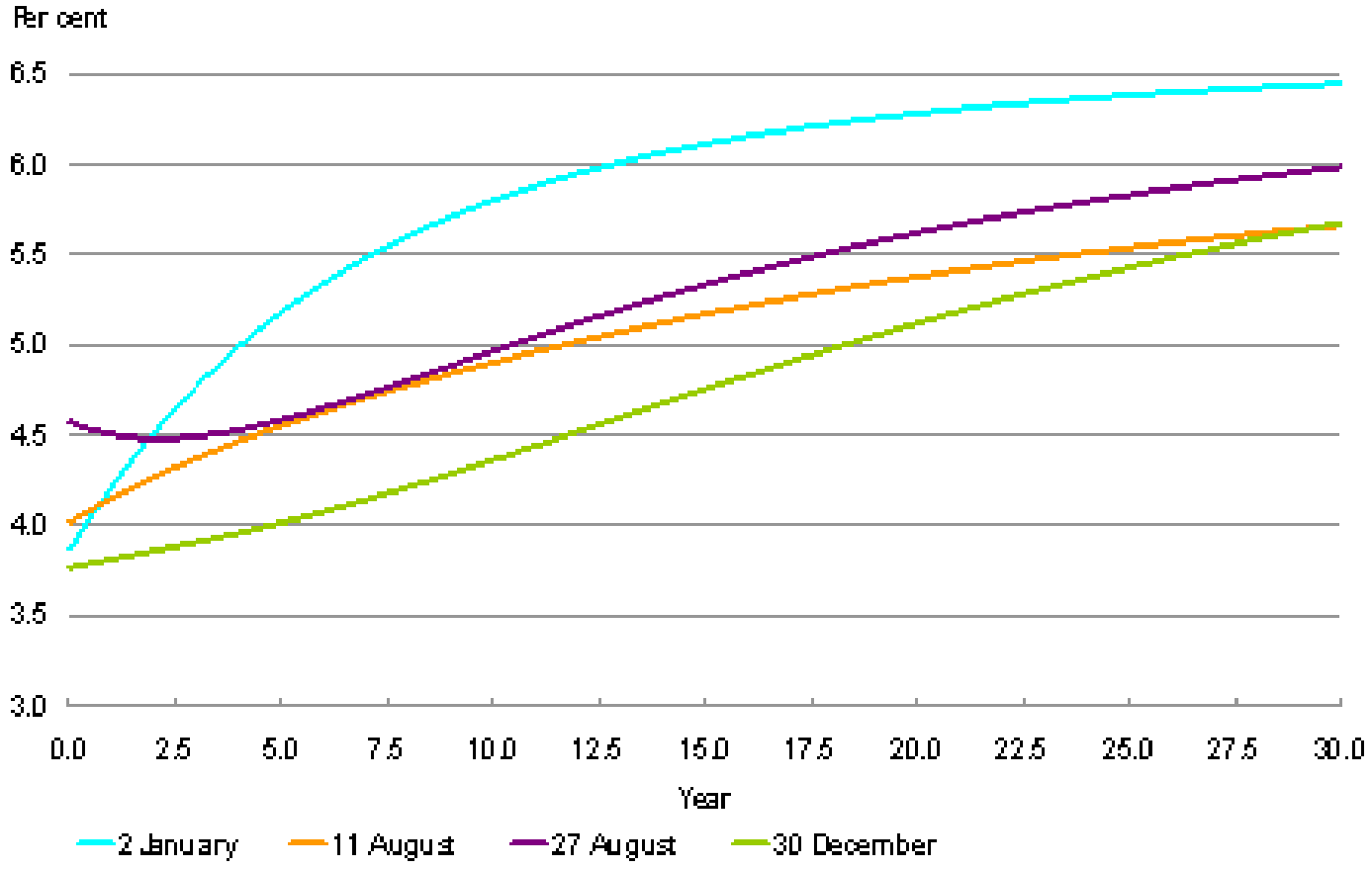
- $L(t,T)$: Simply-compounded spot interest rate

$$L(t,T) = \frac{1-P(t,T)}{P(t,T)(T-t)} \text{ i.e. } P(t,T) = \frac{1}{1+L(t,T)(T-t)} \tag{2}$$

- $Y(t,T)$: Annually-compounded spot interest rate

$$Y(t,T) = \frac{1}{P(t,T)^{1/(T-t)}} - 1 \text{ i.e. } P(t,T) = \frac{1}{(1+Y(t,T))^{(T-t)}} \tag{3}$$

Term structure of interest rates



Reproduced from "Danish Government Borrowing and Debt 1998", Danmarks National Bank, 1999.

Forward interest rates

- Forward-rate agreement : exchange of a fixed-rate payment and a floating-rate payment
- $L(t, T, S)$: Simply-compounded forward interest rate

$$\begin{aligned} L(t, T, S) &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \\ &\text{i.e.} \quad 1 + (S-T)L(t,T,S) = \frac{P(t,T)}{P(t,S)} \end{aligned} \tag{4}$$

- $f(t, T)$: Instantaneous forward interest rate

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \text{ i.e. } P(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \tag{5}$$

Swap rates

- Exchange of fixed-rate cash flows and floating-rate cash flows
- Exchanges at dates $T_{\alpha+1}, \ldots, T_\beta$, with $\tau_i = t_i - T_{i-1}$
- Value at time t of a receiver swap :

$$\Pi^{RS}(t, \alpha, \beta, N, K) = -N(P(t, T_\alpha) - P(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) \tag{6}$$

- Swap rate

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \tag{7}$$

- Link with simple forward rates

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j L(t, T_{j-1}, T_j)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j L(t, T_{j-1}, T_j)}} \tag{8}$$

Caps, floors and swaptions

- Cap : Payer swap in which only positive cash flows are exchanged
- Floor : Receiver swap in which only positive cash flows are exchanged
- Caplet (floorlet) : One-date cap (floor), i.e. contract with payoff at time T_i

$$N\tau_i \left[L(T_{i-1}, T_i) - K\right]^+ . \tag{9}$$

- Swaption : A European payer swaption is an option giving the right to enter a payer swap (α, β) at maturity T, i.e. contract with payoff at time T if $T = T_\alpha$

$$N\left(\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)\left[L(T_\alpha, T_{i-1}, T_i) - K\right]\right)^+ . \tag{10}$$

Part II

Short-rate models

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Model definition

Original paper

Vasicek, O. (1977). "An equilibrium characterization of the term structure", *Journal of Financial Economics*, **5** (2), 177–188.

Dynamics of the short rate

Short rate $r(t)$ follows an Ornstein-Uhlenbeck process

$$dr_t = \kappa[\theta - r_t]dt + \sigma dW_t \tag{11}$$

with κ, θ, σ positive constants.

Dynamics of the short rate

Proposition

In the Vasicek model, the SDE defining the short rate dynamics can be integrated to obtain

$$r(t) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-u)} dW_u \qquad (12)$$

Short rate $r(t)$ is normally distributed conditionally on $\mathcal{F}_s$.

Price of zero-coupon bonds

Proposition

In the Vasicek model, the price of a zero-coupon bond is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{13}$$

with

$$\begin{cases} A(t,T) &= \exp\left[(\theta - \frac{\sigma^2}{2\kappa^2})(B(t,T) - (T-t)) - \frac{\sigma^2}{4\kappa}B(t,T)^2\right] \\ B(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa} \end{cases} \tag{14}$$

Explicit pricing formula.

Continuously-compounded spot rate

Proposition

In the Vasicek model, the continuously-compounded spot rate is written

$$R(t,T) = R_\infty + (r_t - R_\infty)\frac{1-e^{-\kappa(T-t)}}{\kappa(T-t)} + \frac{\sigma^2}{4\kappa^3(T-t)}(1-e^{-\kappa(T-t)})^2 \quad (15)$$

with

$$R_\infty = \theta - \frac{\sigma^2}{2\kappa^2} \quad (16)$$

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Model definition

Original paper

Cox J.C., Ingersoll J.E. and Ross S.A. (1985). "A Theory of the Term Structure of Interest Rates". *Econometrica* **53**, 385–407.

Dynamics of the short rate

Short rate is given by the following SDE

$$dr_t = \kappa[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t \tag{17}$$

with κ, θ, σ positive constants satisfying $\sigma^2 < 2\kappa\theta$.

Dynamics of the short rate

Proposition

In the CIR model, the short rate $r(t)$ follows a noncentral χ^2 distribution

Pricing of zero-coupon bonds

Proposition

In the CIR model, the price of a zero-coupon bond is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{18}$$

with

$$\begin{cases} A(t,T) &= \left[\dfrac{2\gamma e^{\frac{\kappa+\gamma}{2}(T-t)}}{2\gamma + (\kappa+\gamma)(e^{\gamma(T-t)}-1)}\right]^{\frac{2\kappa\theta}{\sigma^2}} \\ B(t,T) &= \dfrac{2(e^{\gamma(T-t)}-1)}{2\gamma + (\kappa+\gamma)(e^{\gamma(T-t)}-1)} \\ \gamma &= \sqrt{\kappa^2 + 2\sigma^2} \end{cases} \tag{19}$$

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Model definition

Original paper

J Hull and A White (1990). "Pricing interest-rate-derivative securities". *The Review of Financial Studies* **3**, 573–592.

Dynamics of the short rate

Short rate is given by the following SDE

$$dr_t = [b(t) - ar_t]dt + \sigma dW_t \tag{20}$$

with a and σ positive constants.

Non-time-homogeneous extension of the Vasicek model.

Model definition

Proposition

This model can exactly fit the term-structure observed on the market by setting

$$b(t) = \frac{\partial f^M}{\partial T}(0,t) + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{21}$$

Dynamics of the short rate

The short rate SDE can then be integrated to obtain :

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u \tag{22}$$

with $\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$.

Price of zero-coupon bonds

Proposition

In the Hull-White model, the price of a zero-coupon bond is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{23}$$

with

$$\begin{cases} A(t,T) &= \dfrac{P^M(0,T)}{P^M(0,t)} \exp\left[B(t,T)f^M(0,t) - \dfrac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right] \\ B(t,T) &= \dfrac{1}{a}(1-e^{-a(T-t)}) \end{cases} \tag{24}$$

Price of options on a zero-coupon bond

Proposition

In the Hull-White model, the price of a european call option, with strike K and maturity T, on a zero-coupon bond of maturity $S > T$, can be written

$$C_{ZC}^{HW} = P(t,S)\mathcal{N}(q_1) - KP(t,T)\mathcal{N}(q_2) \tag{25}$$

with

$$\begin{cases} \sigma_r^T &= \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}B(T,S) \\ q_1 &= \frac{1}{\sigma_r^T}\ln\frac{P(t,S)}{KP(t,T)} + \frac{\sigma_r^T}{2} \\ q_2 &= q_1 - \sigma_r^T \end{cases} \tag{26}$$

Part III

From short rate models to HJM framework

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Motivations

- Empirical studies : correlations of interests rates by maturity

	1M	3M	6M	1A	2A	3A	4A	5A	7A	10A
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1A	0.546	0.539	0.672	1						
2A	0.235	0.224	0.31	0.88	1					
3A	0.246	0.239	0.384	0.808	0.929	1				
4A	0.209	0.202	0.337	0.742	0.881	0.981	1			
5A	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7A	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10A	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

Reproduced from Martellini, Priaulet, 2000.

- Empirical studies : PCA on correlation matrix

	By correlation matrix				
No. of factors	1	2	3	4	5
DEM	67.7	83.9	93.7	97.1	98.5
JPY	75.8	91.2	94.3	96.6	98.1
USD	75.8	85.1	93.0	97.0	98.9

Reproduced from Jamshidian, Zhu, (1997), Finance and Stochastics **1***(1), 43–67.*

- Correlation in one-factor affine term structure models
- Arbitrage in short/long rates models (Dybvig, Ingersoll, Ross, (1996))

A Gaussian two-factor model (I)

Model definition

The short rate $r(t)$ is written

$$r(t) = x(t) + y(t) + \phi(t), \quad r(0) = r_0, \tag{27}$$

where the two factors x and y are solutions of the following SDEs :

$$\begin{cases} dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0, \\ dy(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0, \end{cases} \tag{28}$$

with $d\langle W_1, W_2\rangle_t = \rho$ and ϕ is a *deterministic* function such that $\phi(0) = r_0$.

A Gaussian two-factor model (II)

Price of a zero-coupon bond

In the Gaussian two-factor model, the price $P(t,T)$ at time t of the T-maturity zero-coupon bond can be written

$$P(t,T) = \exp\{-\int_0^T \phi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\} \tag{29}$$

Fitting the observed term structure

The Gaussian two-factor model fits the observed term structure $P^M(0,T)$ if and only if ϕ is written

$$\phi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}(1-e^{aT})^2 + \frac{\eta^2}{2b^2}(1-e^{bT})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{aT})(1-e^{bT}) \tag{30}$$

A Gaussian two-factor model (III)

Pricing of zero-coupon bond

In the Gaussian two-factor model, the price $P(t,T)$ at time t of the T-maturity zero-coupon bond can be written

$$P(t,T) = A(t,T)\exp\{-B_a(t,T)x(t) - B_b(t,T)y(t)\} \tag{31}$$

where

$$\begin{cases} A(t,T) &= \dfrac{P^M(0,T)}{P^M(0,t)}\exp\{\dfrac{1}{2}(V(t,T) - V(0,T) + V(0,t))\}, \\ B_i(t,T) &= \dfrac{1-e^{-i(T-t)}}{i}. \end{cases} \tag{32}$$

First step allowing the modeling of correlations.

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Framework definition

Original paper

Heath D., Jarrow R. and Morton A.(1992). "Bond Pricing and the Term Structure of Interest Rates: A New Methodology". *Econometrica* **60**, 77–105.

Forward dynamics

The instantaneous forward rates dynamics is given by the following SDE:

$$\begin{cases} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW_t, \\ f(0,T) &= f^M(0,T). \end{cases} \tag{33}$$

No arbitrage condition (I)

Dynamics of the zero-coupon bond prices

In a HJM framework, the price of the T-maturity zero-coupon bond is solution of the following SDE:

$$dP(t,T) = P(t,T)\left[r_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right]dt - \sigma^*(t,T)P(t,T)dW_t, \tag{34}$$

where

$$\begin{cases} \alpha^*(t,T) &= \displaystyle\int_t^T \alpha(t,u)du, \\ \sigma^*(t,T) &= \displaystyle\int_t^T \sigma(t,u)du. \end{cases} \tag{35}$$

No arbitrage condition (II)

Dynamics of the zero-coupon bond prices

In a HJM framework, there is no arbitrage if there exists a process $(\theta_t)_{0 \leq t \leq \bar{T}}$ satisfying

$$\alpha(t, T) = \sigma(t, T)\left[\sigma^*(t, T) + \theta(t)\right] \tag{36}$$

In this case, dynamics in the model can be rewritten under a risk-neutral measure $\mathbf{Q}$:

$$\begin{array}{rclcl} df(t, T) & = & \sigma(t, T)\sigma^*(t, T)dt & + & \sigma(t, T)dW_t^{\mathbf{Q}} \\ dP(t, T) & = & r_t P(t, T)dt & - & \sigma^*(t, T)P(t, T)dW_t^{\mathbf{Q}} \end{array} \tag{37}$$

Markovianity in the HJM framework (I)

Dynamics of the short rate

In a no arbitrage HJM framework, the short rate can be written

$$r(t) = f(0,t) + \int_0^t \sigma(u,t) \int_u^t \sigma(u,s)ds\, du + \int_0^t \sigma(u,t)dW_u \qquad (38)$$

Choice of volatilities to get a markovian model:

- Separation of variables : $\sigma(t,T) = \xi(t)\phi(T)$
- Ritchken and Sankarasubramanian (1995) : $\sigma(t,T) = \eta(t)e^{-\int_t^T \kappa(u)du}$

Markovianity in the HJM framework (II)

Ritchken and Sankarasubramanian volatility (1995)

In a 1D HJM framework with $\sigma(t, T)$ T-differentiable, every derivative product is completely determined by a two-dimensional Markov process if and only if $\sigma(t, T) = \eta(t)e^{-\int_t^T \kappa(u)du}$ where η is an adapted process and κ is a deterministic and integrable process.

Link with affine models

Proposition

If the SDE $dr_t = b(t, r_t)dt + \gamma(t, r_t)dW_t$ defines a short rate model with an affine term structure $P(t, T) = A(t, T)e^{-B(t,T)r(t)}$, then this model belongs to the HJM framework with $\sigma(t, T) = \frac{\partial}{\partial T}B(t, T)\gamma(t, r_t)$.

One can check that the Vasicek, CIR and Hull-White models are one-dimensional no-arbitrage HJM models.

Choices of volatility

Ho-Lee

$$\sigma(t, T) = \sigma \quad (\text{constant}) \tag{39}$$

Vasicek/Hull-White

$$\sigma(t, T) = \gamma(t) e^{-\lambda(T-t)} \tag{40}$$

Pricing of caplet

Proposition

In a no-arbitrage HJM framework, the price of a caplet of maturity T, strike K, paid in $T+\theta$ is written :

$$C(t,T,K,\theta) = P(t,T+\theta)\left[(1+\theta L(t,T,T+\theta)\mathcal{N}(d_1) - (1+\theta K)\mathcal{N}(d_0)\right] \tag{41}$$

where

$$\begin{cases} d_0 &= \dfrac{1}{\Sigma(t,T)} \ln \dfrac{1+\theta L(t,T,T+\theta)}{1+\theta K} - \dfrac{1}{2}\Sigma^2(t,T) \\ d_1 &= d_0 + \Sigma(t,T) \\ \Sigma(t,T) &= \displaystyle\int_t^T (\sigma^*(u,T+\theta) - \sigma^*(u,T))^2 du \end{cases} \tag{42}$$

Part IV

Libor market models

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General change of measure

Theorem

Assume there exists a numeraire $(M_t)_{t\geq 0}$ and an equivalent measure $\mathbf{Q}^M$ such that the price of any traded asset X "discounted" by the process M is a $\mathbf{Q}^M$-martingale, i.e. $\frac{X_t}{M_t} = E^{\mathbf{Q}^M}\left[\frac{X_T}{M_T}\middle|\mathcal{F}_t\right]$.
Let $(N_t)_{t\geq 0}$ be a numeraire.
Then there exists an equivalent probability measure $\mathbf{Q}^N$ such that the price of any traded asset X "discounted" by N is a $\mathbf{Q}^N$-martingale, i.e. $\frac{X_t}{N_t} = E^{\mathbf{Q}^N}\left[\frac{X_T}{N_T}\middle|\mathcal{F}_t\right]$.

$\mathbf{Q}^N$ is defined by the Radon-Nikodym derivative

$$\left.\frac{d\mathbf{Q}^N}{d\mathbf{Q}^M}\right|_{\mathcal{F}_T} = \frac{N_T}{N_0}\frac{M_0}{M_T}. \tag{43}$$

Changing from risk-neutral measure

Proposition

Let $\mathbf{Q}$ be the risk-neutral measure associated with a riskless numeraire $\beta(t) = e^{\int_0^t r_u du}$. Let X be a traded asset with $\mathbf{Q}$-dynamics

$$dX_t = r_t X_t dt + \sigma^X(t, X_t) dW_t^{\mathbf{Q}} \tag{44}$$

Let N be another traded asset :

$$dN_t = r_t N_t dt + \sigma^N(t, N_t) dW_t^{\mathbf{Q}} \tag{45}$$

Then $\frac{X_t}{N_t}$ is a $\mathbf{Q}^N$-martingale with dynamics :

$$d\left(\frac{X_t}{N_t}\right) = \frac{X_t}{N_t}\left(\sigma^X(t, X_t) - \sigma^N(t, N_t)\right)\sigma^N(t, N_t) dW_t^{\mathbf{Q}^N} \tag{46}$$

where $dW_t^{\mathbf{Q}^N} = dW_t^{\mathbf{Q}} - \sigma^N(t, N_t)dt$ is a $\mathbf{Q}^N$-brownian motion.

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The Black formula

Proposition

The Black formula for a caplet (maturity T, strike K) on the θ-tenor Libor $L(.,.+\theta$ and paying at date $T+\theta$ is written :

$$C(t,T,K,\theta) = P(t,T+\theta)\left[L(t,T,T+\theta)\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] \tag{47}$$

where

$$\left\{\begin{array}{lcl} d_1 & = & \dfrac{1}{\sigma\sqrt{T-t}}\ln\dfrac{L(t,T,T+\theta)}{K} + \dfrac{1}{2}\sigma\sqrt{T-t} \\ d_2 & = & d_1 - \sigma\sqrt{T-t} \end{array}\right. \tag{48}$$

When is this formula justified ?

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Model definition (I)

Original paper

Brace A., Gatarek D. and Musiela M. (1997). "The Market Model of Interest Rate Dynamics", *Mathematical Finance*, **7**, 127–155.

Assumptions ans notations

- Calendar $0 < T_0 < T_1 < \ldots < T_M$.
- M forward Libor rates $(L(t, T_0, T_1), \ldots, L(t, T_{M-1}, T_M)$ with tenor $\theta_i = T_i - T_{i-1}$.
- Notation : $\forall i = 1, \ldots, M, L_i(t) = L(t, T_{i-1}, T_i)$

Model definition (II)

Dynamics of the forward Libor rates

Each forward Libor is assumed to be a martingale with respect to the associated forward measure :

$$\frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW^{i,\mathbf{Q}^{T_i}}(t) \tag{49}$$

where $\gamma_i(t)$ is a *deterministic* function.

No arbitrage condition

Proposition

In the BGM model, the no arbitrage condition gives the following relationship between the volatilities γ_i of the forward Libor and the volatilities Γ_i of the zero-coupon bonds $P(t, T_i)$:

$$\gamma_i(t) = \frac{1 + \theta_i L_i(t)}{\theta_i L_i(t)} \left[\Gamma_i(t) - \Gamma_{i-1}(t)\right]. \tag{50}$$

Pricing caplets

Proposition

In the BGM model, the price at time 0 of a post-paid caplet (strike K, maturity T_{i-1} on a Libor rate $L(T_{i-1}, T_i)$ is given by:

$$C(0, T_{i-1}, K, \theta_i) = P(0, T_i)\left[L_i(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] \tag{51}$$

where

$$\begin{cases} d_1 &= \dfrac{1}{v}\ln\dfrac{L_i(0)}{K} + \dfrac{1}{2}v \\ d_2 &= d_1 - v \\ v &= \int_0^{T_{i-1}} \gamma_i^2(t)dt \end{cases} \tag{52}$$

The volatility implied by the Black formula is then

$$\sigma_{\text{imp}}^{\text{Black}}(L_i) = \sqrt{\frac{1}{T_{i-1}}\int_0^{T_{i-1}} \gamma_i^2(t)dt} \tag{53}$$

Specifying Libor volatilities (I)

Simple choice : constant volatilities

$$\forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_i \text{ constant.} \tag{54}$$

	$[0, T_0]$	$]T_0, T_1]$	$]T_1, T_2]$	$\ldots$	$]T_{M-2}, T_{M-1}]$
$L_1(t)$	γ_1	dead	dead	$\ldots$	dead
$L_2(t)$	γ_2	γ_2	dead	$\ldots$	dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$L_M(t)$	γ_M	γ_M	γ_M	$\ldots$	γ_M

Specifying Libor volatilities (II)

Another simple choice : piecewise-constant volatilities

$$\forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_{i,\beta(t)} \text{ constant.} \tag{55}$$

	$[0, T_0]$	$]T_0, T_1]$	$]T_1, T_2]$	$\ldots$	$]T_{M-2}, T_{M-1}]$
$L_1(t)$	$\gamma_{1,1}$	dead	dead	$\ldots$	dead
$L_2(t)$	$\gamma_{2,1}$	$\gamma_{2,2}$	dead	$\ldots$	dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$L_M(t)$	$\gamma_{M,1}$	$\gamma_{M,2}$	$\gamma_{M,3}$	$\ldots$	$\gamma_{M,M}$

Specifying Libor volatilities (III)

Simpler : piecewise-constant volatility that depends only on the time to maturity

$$\forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_{i,\beta(t)} = \eta_{i-(\beta(t)-1)} \text{ constant.} \tag{56}$$

	$[0, T_0]$	$]T_0, T_1]$	$]T_1, T_2]$	$\ldots$	$]T_{M-2}, T_{M-1}]$
$L_1(t)$	η_1	dead	dead	$\ldots$	dead
$L_2(t)$	η_2	η_1	dead	$\ldots$	dead
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$L_M(t)$	η_M	η_{M-1}	η_{M-2}	$\ldots$	η_1

Specifying Libor volatilities (IV)

Parametric choices

One may also define Libor volatilities with $\forall i = 1, \ldots, M$, :

$$\gamma_i(t) = [a(T_{i-1} - t) + d]\, e^{-b(T_{i-1}-t)} + c \tag{57}$$

or

$$\gamma_i(t) = \eta_i \left[[a(T_{i-1} - t) + d]\, e^{-b(T_{i-1}-t)} + c \right] \tag{58}$$

Dynamics of the forward Libor rates under a unique forward measure

Proposition

Let $i \in \{1, \ldots, M\}$. The dynamics of the forward Libor rates $L_i(t)$ under the forward measure $\mathbf{Q}^{T_k}, k = 1, \ldots, M$ is given by the following SDE:

$$\text{if } k < i, \quad \frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW^{i,\mathbf{Q}^{T_k}} - \sum_{j=k}^{i} \rho_{ij} \gamma_i(t) \gamma_j(t) \frac{\theta_j L_j(t)}{1 + \theta_j L_j(t)} dt,$$

$$\text{if } k = i, \quad \frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW^{i,\mathbf{Q}^{T_i}}$$

$$\text{if } k > i, \quad \frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW^{i,\mathbf{Q}^{T_k}} + \sum_{j=i+1}^{k} \rho_{ij} \gamma_i(t) \gamma_j(t) \frac{\theta_j L_j(t)}{1 + \theta_j L_j(t)} dt.$$

Introducing the spot Libor measure

Definition

The spot Libor numeraire is defined as :

$$B(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)} \tag{59}$$

Proposition

Under the spot Libor measure $\mathbf{Q}^B$ associated with the numeraire $B(t)$, the dynamics of the foward Libor $L_i(t)$ is written:

$$\frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW^{\mathbf{Q}^B} + \sum_{j=\beta(t)}^{i} \rho_{ij}\gamma_i(t)\gamma_j(t)\frac{\theta_j L_j(t)}{1+\theta_j L_j(t)} dt. \tag{60}$$

Swap market model

Original paper

Jamshidian F. (1997). "LIBOR and swap market models and measures", *Finance and Stochastics*, **1** (4), 293–330.)

Model definition

A swap market models assumes that the swap rate $S_{\alpha,\beta}$ is solution of the SDE :

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = \gamma_{\alpha,\beta}(t) dW^{\mathbf{Q}^{\alpha,\beta}}(t), \tag{61}$$

where $\mathbf{Q}^{\alpha,\beta}$ is the measure linked with numeraire $\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$.

- Compatibility with the Black formula for swaption
- Theoretical inconsistency with the BGM market model

Other LIBOR approaches

- Markov-functional Libor models (Hunt P., Kennedy J. and Pelsser A. (2000) " Markov-functional interest rate models ", *Finance and Stochastics*, **4** (4), 391–408.)
- Affine Libor Models (Keller-Ressel M., Papapantoleon A., Teichmann J. (2009). "A new approach to LIBOR modeling")