Interest rates stochastic models

Ioane Muni Toke

Ecole Centrale Paris
Option Mathématiques Appliquées
Majeure Mathématiques Financières

December 2010 - January 2011
Course outline

Lecture 1  Basic concepts and short rate models
Lecture 2  From short rate models to the HJM framework
Lecture 3  Libor Market Models
Lecture 4  Practical aspects of market models - I
            (E.Durand, Société Générale)
Lecture 5  Practical aspects of market models - II
            (E.Durand, Société Générale)
Useful bibliography

This short course uses material from:

- Original research papers (references below).
Part I

Basic concepts
Spot interest rates

- \( r(t) \): Instantaneous (interbank) rate, or short rate.
- \( P(t, T) \): Price at time \( t \) of a \( T \)-maturity zero-coupon bond
- \( R(t, T) \): Continuously-compounded spot interest rate

\[
R(t, T) = -\frac{\ln P(t, T)}{T - t} \quad \text{i.e.} \quad P(t, T) = e^{-R(t, T)(T-t)} \quad (1)
\]

- \( L(t, T) \): Simply-compounded spot interest rate

\[
L(t, T) = \frac{1 - P(t, T)}{P(t, T)(T - t)} \quad \text{i.e.} \quad P(t, T) = \frac{1}{1 + L(t, T)(T - t)} \quad (2)
\]

- \( Y(t, T) \): Annually-compounded spot interest rate

\[
Y(t, T) = \frac{1}{P(t, T)^{1/(T-t)}} - 1 \quad \text{i.e.} \quad P(t, T) = \frac{1}{(1 + Y(t, T))^{(T-t)}} \quad (3)
\]
Term structure of interest rates

Forward interest rates

- Forward-rate agreement: exchange of a fixed-rate payment and a floating-rate payment
- \( L(t, T, S) \): Simply-compounded forward interest rate

\[
L(t, T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right)
\]

i.e. \( 1 + (S - T)L(t, T, S) = \frac{P(t, T)}{P(t, S)} \) \( (4) \)

- \( f(t, T) \): Instantaneous forward interest rate

\[
f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T}
\]
i.e. \( P(t, T) = \exp \left( -\int_t^T f(t, u)du \right) \) \( (5) \)
Swap rates

- Exchange of fixed-rate cash flows and floating-rate cash flows
- Exchanges at dates $T_{\alpha+1}, \ldots, T_\beta$, with $\tau_i = t_i - T_{i-1}$
- Value at time $t$ of a receiver swap:

\[
\Pi^{RS}(t, \alpha, \beta, N, K) = -N(P(t, T\alpha) - P(t, T\beta)) + N \sum_{i=\alpha+1}^{\beta} \tau_i KP(t, T_i)
\]  
(6)

- Swap rate

\[
S_{\alpha,\beta}(t) = \frac{P(t, T\alpha) - P(t, T\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}
\]  
(7)

- Link with simple forward rates

\[
S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j L(t, T_{j-1}, T_j)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j L(t, T_{j-1}, T_j)}}
\]  
(8)
Caps, floors and swaptions

- **Cap**: Payer swap in which only positive cash flows are exchanged
- **Floor**: Receiver swap in which only positive cash flows are exchanged
- **Caplet (floorlet)**: One-date cap (floor), i.e. contract with payoff at time $T_i$
  \[ N_{T_i} [L(T_{i-1}, T_i) - K]^+. \]  
  \[ (9) \]
- **Swaption**: A European payer swaption is an option giving the right to enter a payer swap $(\alpha, \beta)$ at maturity $T$, i.e. contract with payoff at time $T$ if $T = T_\alpha$
  \[ N \left( \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) [L(T_\alpha, T_{i-1}, T_i) - K] \right)^+. \]  
  \[ (10) \]
Part II

Short-rate models
Table of contents

1. The Vasicek model
2. The CIR model
3. The Hull-White (extended Vasicek) model
Table of contents

1. The Vasicek model
2. The CIR model
3. The Hull-White (extended Vasicek) model
The Vasicek model

Model definition

Original paper


Dynamics of the short rate

Short rate $r(t)$ follows an Ornstein-Uhlenbeck process

$$dr_t = \kappa[\theta - r_t]dt + \sigma dW_t$$

with $\kappa, \theta, \sigma$ positive constants.
Dynamics of the short rate

**Proposition**

In the Vasicek model, the SDE defining the short rate dynamics can be integrated to obtain

\[ r(t) = r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \sigma \int_s^t e^{-\kappa(t-u)} dW_u \]  

(12)

Short rate \( r(t) \) is normally distributed conditionally on \( \mathcal{F}_s \).
Price of zero-coupon bonds

**Proposition**

In the Vasicek model, the price of a zero-coupon bond is given by

\[ P(t, T) = A(t, T) e^{-B(t, T)r(t)} \]  \hspace{1cm} (13)

with

\[
\begin{align*}
A(t, T) &= \exp \left[ (\theta - \frac{\sigma^2}{2\kappa^2})(B(t, T) - (T - t)) - \frac{\sigma^2}{4\kappa} B(t, T)^2 \right] \\
B(t, T) &= \frac{1 - e^{-\kappa(T-t)}}{\kappa}
\end{align*}
\]  \hspace{1cm} (14)

Explicit pricing formula.
Proposition

In the Vasicek model, the continuously-compounded spot rate is written

$$R(t, T) = R_\infty + (r_t - R_\infty) \frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)} + \frac{\sigma^2}{4\kappa^3(T-t)}(1 - e^{-\kappa(T-t)})^2$$  \hspace{1cm} (15)$$

with

$$R_\infty = \theta - \frac{\sigma^2}{2\kappa^2}$$  \hspace{1cm} (16)$$
Table of contents

1. The Vasicek model

2. The CIR model

3. The Hull-White (extended Vasicek) model
Model definition

Original paper


Dynamics of the short rate

Short rate is given by the following SDE

\[ dr_t = \kappa [\theta - r_t] dt + \sigma \sqrt{r_t} dW_t \]  

(17)

with \( \kappa, \theta, \sigma \) positive constants satisfying \( \sigma^2 < 2\kappa\theta \).
The CIR model

Dynamics of the short rate

Proposition

In the CIR model, the short rate $r(t)$ follows a noncentral $\chi^2$ distribution.
Pricing of zero-coupon bonds

Proposition
In the CIR model, the price of a zero-coupon bond is

\[ P(t, T) = A(t, T)e^{-B(t,T)r(t)} \]  \hspace{1cm} (18)

with

\[ A(t, T) = \begin{bmatrix} 2\gamma e^{\frac{\kappa + \gamma}{2}(T-t)} \\ 2\gamma + (\kappa + \gamma)(e^{\gamma(T-t)} - 1) \end{bmatrix} \frac{2\kappa \theta}{\sigma^2} \]

\[ B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{2\gamma + (\kappa + \gamma)(e^{\gamma(T-t)} - 1)} \]

\[ \gamma = \sqrt{\kappa^2 + 2\sigma^2} \]  \hspace{1cm} (19)
Table of contents

1. The Vasicek model
2. The CIR model
3. The Hull-White (extended Vasicek) model
Model definition

Original paper


Dynamics of the short rate

Short rate is given by the following SDE

\[ dr_t = [b(t) - ar_t]dt + \sigma dW_t \]  

(20)

with \(a\) and \(\sigma\) positive constants.

Non-time-homogeneous extension of the Vasicek model.
The Hull-White (extended Vasicek) model

Model definition

Proposition
This model can exactly fit the term-structure observed on the market by setting

\[
b(t) = \frac{\partial f^M}{\partial T}(0, t) + af^M(0, t) + \frac{\sigma^2}{2a} \left( 1 - e^{-2at} \right) \tag{21}\]

Dynamics of the short rate

The short rate SDE can then be integrated to obtain:

\[
r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u \tag{22}\]

with \(\alpha(t) = f^M(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2.\)
Price of zero-coupon bonds

**Proposition**

In the Hull-White model, the price of a zero-coupon bond is given by

\[ P(t, T) = A(t, T)e^{-B(t,T)r(t)} \] (23)

with

\[
\begin{align*}
A(t, T) &= \frac{P^M(0, T)}{P^M(0, t)} \exp \left[ B(t, T)f^M(0, t) - \frac{\sigma^2}{4a} (1 - e^{-2at})B(t, T)^2 \right] \\
B(t, T) &= \frac{1}{a} (1 - e^{-a(T-t)})
\end{align*}
\] (24)
Price of options on a zero-coupon bond

Proposition

In the Hull-White model, the price of a european call option, with strike $K$ and maturity $T$, on a zero-coupon bond of maturity $S > T$, can be written

$$C^{HW}_{ZC} = P(t, S)N(q_1) - KP(t, T)N(q_2)$$

(25)

with

$$\begin{cases}
\sigma_r^T &= \sigma \sqrt{\frac{1-e^{-2a(T-t)}}{2a}} B(T, S) \\
q_1 &= \frac{1}{\sigma_r^T} \ln \frac{P(t, S)}{KP(t, T)} + \frac{\sigma_r^T}{2} \\
q_2 &= q_1 - \sigma_r^T
\end{cases}$$

(26)
Part III

From short rate models to HJM framework
Table of contents

4 Multifactor models

5 The HJM framework
Table of contents

4 Multifactor models

5 The HJM framework
Motivations

- **Empirical studies**: correlations of interests rates by maturity
  
  
<table>
<thead>
<tr>
<th></th>
<th>1M</th>
<th>3M</th>
<th>6M</th>
<th>1A</th>
<th>2A</th>
<th>3A</th>
<th>4A</th>
<th>5A</th>
<th>7A</th>
<th>10A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1M</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3M</td>
<td>0.999</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6M</td>
<td>0.908</td>
<td>0.914</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1A</td>
<td>0.546</td>
<td>0.539</td>
<td>0.672</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2A</td>
<td>0.235</td>
<td>0.224</td>
<td>0.31</td>
<td>0.88</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3A</td>
<td>0.246</td>
<td>0.239</td>
<td>0.384</td>
<td>0.808</td>
<td>0.929</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4A</td>
<td>0.209</td>
<td>0.202</td>
<td>0.337</td>
<td>0.742</td>
<td>0.881</td>
<td>0.981</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5A</td>
<td>0.163</td>
<td>0.154</td>
<td>0.255</td>
<td>0.7</td>
<td>0.859</td>
<td>0.936</td>
<td>0.981</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7A</td>
<td>0.107</td>
<td>0.097</td>
<td>0.182</td>
<td>0.617</td>
<td>0.792</td>
<td>0.867</td>
<td>0.927</td>
<td>0.97</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10A</td>
<td>0.073</td>
<td>0.063</td>
<td>0.134</td>
<td>0.549</td>
<td>0.735</td>
<td>0.811</td>
<td>0.871</td>
<td>0.917</td>
<td>0.966</td>
<td>1</td>
</tr>
</tbody>
</table>


- **Empirical studies**: PCA on correlation matrix

  |
  No. of factors | 1  | 2  | 3  | 4  | 5  |
  1              | 67.7| 83.9| 93.7| 97.1| 98.5|
  JPY            | 75.8| 91.2| 94.3| 96.6| 98.1|
  USD            | 75.8| 85.1| 93.0| 97.0| 98.9|


- Correlation in one-factor affine term structure models

- Arbitrage in short/long rates models (Dybvig, Ingersoll, Ross, (1996))
A Gaussian two-factor model (I)

Model definition

The short rate $r(t)$ is written

$$r(t) = x(t) + y(t) + \phi(t), \quad r(0) = r_0,$$

(27)

where the two factors $x$ and $y$ are solutions of the following SDEs:

$$\begin{align*}
    dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0, \\
    dy(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0,
\end{align*}$$

(28)

with $d\langle W_1, W_2 \rangle_t = \rho$ and $\phi$ is a deterministic function such that $\phi(0) = r_0$. 
A Gaussian two-factor model (II)

Price of a zero-coupon bond

In the Gaussian two-factor model, the price $P(t, T)$ at time $t$ of the $T$-maturity zero-coupon bond can be written

$$P(t, T) = \exp\{-\int_0^T \phi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t, T)\}$$ (29)

Fitting the observed term structure

The Gaussian two-factor model fits the observed term structure $P^M(0, T)$ if and only if $\phi$ is written

$$\phi(T) = f^M(0, T) + \frac{\sigma^2}{2a^2}(1-e^{aT})^2 + \frac{\eta^2}{2b^2}(1-e^{bT})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{aT})(1-e^{bT})$$ (30)
Pricing of zero-coupon bond

In the Gaussian two-factor model, the price $P(t, T)$ at time $t$ of the $T$-maturity zero-coupon bond can be written

$$P(t, T) = A(t, T) \exp\{ -B_a(t, T)x(t) - B_b(t, T)y(t) \}$$  \hspace{1cm} (31)

where

$$\begin{align*}
A(t, T) &= \frac{P^M(0, T)}{P^M(0, t)} \exp\{ \frac{1}{2}(V(t, T) - V(0, T) + V(0, t)) \}, \\
B_i(t, T) &= \frac{1 - e^{-i(T-t)}}{i}.
\end{align*}$$  \hspace{1cm} (32)

First step allowing the modeling of correlations.
Table of contents

4 Multifactor models

5 The HJM framework
Framework definition

Original paper

Forward dynamics
The instantaneous forward rates dynamics is given by the following SDE:

\[
\begin{align*}
    df(t, T) &= \alpha(t, T)dt + \sigma(t, T)dW_t, \\
    f(0, T) &= f^M(0, T).
\end{align*}
\]  

(33)
In a HJM framework, the price of the $T$-maturity zero-coupon bond is solution of the following SDE:

$$dP(t, T) = P(t, T) \left[ r_t - \alpha^*(t, T) + \frac{1}{2} \sigma^*(t, T)^2 \right] dt - \sigma^*(t, T)P(t, T)dW_t,$$

(34)

where

$$\begin{align*}
\alpha^*(t, T) &= \int_t^T \alpha(t, u)du, \\
\sigma^*(t, T) &= \int_t^T \sigma(t, u)du.
\end{align*}$$

(35)
The HJM framework

No arbitrage condition (II)

Dynamics of the zero-coupon bond prices

In a HJM framework, there is no arbitrage if there exists a process 
\( (\theta_t)_{0 \leq t \leq \bar{T}} \) satisfying

\[
\alpha(t, T) = \sigma(t, T)[\sigma^*(t, T) + \theta(t)]
\]  \hspace{1cm} (36)

In this case, dynamics in the model can be rewritten under a risk-neutral measure \( Q \):

\[
df(t, T) = \sigma(t, T)\sigma^*(t, T)dt + \sigma(t, T)dW^Q_t
\]
\[
dP(t, T) = r_tP(t, T)dt - \sigma^*(t, T)P(t, T)dW^Q_t
\]  \hspace{1cm} (37)
Dynamics of the short rate

In a no arbitrage HJM framework, the short rate can be written

\[ r(t) = f(0, t) + \int_0^t \sigma(u, t) \int_u^t \sigma(u, s) ds \, du + \int_0^t \sigma(u, t) dW_u \]  

Choice of volatilities to get a markovian model:

- Separation of variables: \( \sigma(t, T) = \xi(t)\phi(T) \)
- Ritchken and Sankarasubramanian (1995): \( \sigma(t, T) = \eta(t)e^{-\int_t^T \kappa(u) du} \)
Ritchken and Sankarasubramanian volatility (1995)

In a 1D HJM framework with \( \sigma(t, T) \) \( T \)-differentiable, every derivative product is completely determined by a two-dimensional Markov process if and only if

\[
\sigma(t, T) = \eta(t) e^{-\int_t^T \kappa(u) du} \]

where \( \eta \) is an adapted process and \( \kappa \) is a deterministic and integrable process.
The HJM framework

Link with affine models

Proposition

If the SDE $dr_t = b(t, r_t)dt + \gamma(t, r_t)dW_t$ defines a short rate model with an affine term structure $P(t, T) = A(t, T)e^{-B(t, T)r(t)}$, then this model belongs to the HJM framework with $\sigma(t, T) = \frac{\partial}{\partial T} B(t, T) \gamma(t, r_t)$.

One can check that the Vasicek, CIR and Hull-White models are one-dimensional no-arbitrage HJM models.
# Choices of volatility

## Ho-Lee

\[ \sigma(t, T) = \sigma \quad \text{(constant)} \]  

(39)

## Vasicek/Hull-White

\[ \sigma(t, T) = \gamma(t)e^{-\lambda(T-t)} \]  

(40)
Pricing of caplet

Proposition

In a no-arbitrage HJM framework, the price of a caplet of maturity $T$, strike $K$, paid in $T + \theta$ is written:

$$C(t, T, K, \theta) = P(t, T + \theta) \left[ (1 + \theta L(t, T, T + \theta) N(d_1) - (1 + \theta K) N(d_0) \right]$$

(41)

where

$$\begin{align*}
    d_0 &= \frac{1}{\Sigma(t, T)} \ln \frac{1 + \theta L(t, T, T + \theta)}{1 + \theta K} - \frac{1}{2} \Sigma^2(t, T) \\
    d_1 &= d_0 + \Sigma(t, T) \\
    \Sigma(t, T) &= \int_t^T (\sigma^*(u, T + \theta) - \sigma^*(u, T))^2 du
\end{align*}$$

(42)
Part IV

Libor market models
Table of contents

6 Change of numeraire

7 The Black Formula

8 The BGM market model
Table of contents

6 Change of numeraire

7 The Black Formula

8 The BGM market model
General change of measure

**Theorem**

Assume there exists a numeraire \((M_t)_{t \geq 0}\) and an equivalent measure \(Q^M\) such that the price of any traded asset \(X\) “discounted” by the process \(M\) is a \(Q^M\)-martingale, i.e.

\[
\frac{X_t}{M_t} = E^{Q^M} \left[ \frac{X_T}{M_T} \bigg| \mathcal{F}_t \right].
\]

Let \((N_t)_{t \geq 0}\) be a numeraire. Then there exists an equivalent probability measure \(Q^N\) such that the price of any traded asset \(X\) “discounted” by \(N\) is a \(Q^N\)-martingale, i.e.

\[
\frac{X_t}{N_t} = E^{Q^N} \left[ \frac{X_T}{N_T} \bigg| \mathcal{F}_t \right].
\]

\(Q^N\) is defined by the Radon-Nikodym derivative

\[
\frac{dQ^N}{dQ^M} \bigg|_{\mathcal{F}_T} = \frac{N_T}{N_0} \frac{M_0}{M_T}.
\]

(43)
Proposition

Let $Q$ be the risk-neutral measure associated with a riskless numeraire $\beta(t) = e^{\int_0^t r_u du}$. Let $X$ be a traded asset with $Q$-dynamics

$$dX_t = r_t X_t dt + \sigma^X(t, X_t) dW_t^Q$$

(44)

Let $N$ be another traded asset:

$$dN_t = r_t N_t dt + \sigma^N(t, N_t) dW_t^Q$$

(45)

Then $\frac{X_t}{N_t}$ is a $Q^N$-martingale with dynamics:

$$d \left( \frac{X_t}{N_t} \right) = \frac{X_t}{N_t} \left( \sigma^X(t, X_t) - \sigma^N(t, N_t) \right) \sigma^N(t, N_t) dW_t^{Q^N}$$

(46)

where $dW_t^{Q^N} = dW_t^Q - \sigma^N(t, N_t) dt$ is a $Q^N$-brownian motion.
Table of contents

6 Change of numeraire

7 The Black Formula

8 The BGM market model
The Black formula

**Proposition**

The Black formula for a caplet (maturity $T$, strike $K$) on the $\theta$-tenor Libor $L(., . + \theta$ and paying at date $T + \theta$ is written:

$$C(t, T, K, \theta) = P(t, T + \theta) [L(t, T, T + \theta)N(d_1) - KN(d_2)]$$  \hspace{1cm} (47)

where

$$
\begin{align*}
  d_1 &= \frac{1}{\sigma \sqrt{T - t}} \ln \frac{L(t, T, T + \theta)}{K} + \frac{1}{2} \sigma \sqrt{T - t} \\
  d_2 &= d_1 - \sigma \sqrt{T - t}
\end{align*}
$$  \hspace{1cm} (48)

When is this formula justified?
Table of contents

6 Change of numeraire

7 The Black Formula

8 The BGM market model
Model definition (I)

Original paper


Assumptions ans notations

- Calendar $0 < T_0 < T_1 < \ldots < T_M$.
- $M$ forward Libor rates $(L(t, T_0, T_1), \ldots, L(t, T_{M-1}, T_M)$ with tenor $\theta_i = T_i - T_{i-1}$.
- Notation : $\forall i = 1, \ldots, M, L_i(t) = L(t, T_{i-1}, T_i)$
Dynamics of the forward Libor rates

Each forward Libor is assumed to be a martingale with respect to the associated forward measure:

\[
\frac{dL_i(t)}{L_i(t)} = \gamma_i(t)dW_{i,T_i}^{i,Q}(t)
\]  (49)

where \(\gamma_i(t)\) is a \textit{deterministic} function.
No arbitrage condition

Proposition

In the BGM model, the no arbitrage condition gives the following relationship between the volatilities $\gamma_i$ of the forward Libor and the volatilities $\Gamma_i$ of the zero-coupon bonds $P(t, T_i)$:

$$\gamma_i(t) = \frac{1 + \theta_i L_i(t)}{\theta_i L_i(t)} \left[ \Gamma_i(t) - \Gamma_{i-1}(t) \right].$$

(50)
Pricing caplets

Proposition

In the BGM model, the price at time 0 of a post-paid caplet (strike $K$, maturity $T_{i-1}$ on a Libor rate $L(T_{i-1}, T_i)$) is given by:

$$C(0, T_{i-1}, K, \theta_i) = P(0, T_i) [L_i(0)N(d_1) - K N(d_2)]$$

(51)

where

$$d_1 = \frac{1}{\nu} \ln \frac{L_i(0)}{K} + \frac{1}{2} \nu$$

$$d_2 = d_1 - \nu$$

$$\nu = \int_0^{T_{i-1}} \gamma_i^2(t) dt$$

(52)

The volatility implied by the Black formula is then

$$\sigma_{\text{imp}}^{\text{Black}}(L_i) = \sqrt{\frac{1}{T_{i-1}} \int_0^{T_{i-1}} \gamma_i^2(t) dt}$$

(53)
Specifying Libor volatilities (I)

Simple choice: constant volatilities

\[ \forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_i \text{ constant}. \quad (54) \]

<table>
<thead>
<tr>
<th>( L_1(t) )</th>
<th>([0, T_0])</th>
<th>( T_0, T_1 )</th>
<th>( T_1, T_2 )</th>
<th>( \ldots )</th>
<th>( T_{M-2}, T_{M-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_1 )</td>
<td>dead</td>
<td>dead</td>
<td>( \ldots )</td>
<td>dead</td>
<td></td>
</tr>
<tr>
<td>( L_2(t) )</td>
<td>( \gamma_2 )</td>
<td>( \gamma_2 )</td>
<td>dead</td>
<td>( \ldots )</td>
<td>dead</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( L_M(t) )</td>
<td>( \gamma_M )</td>
<td>( \gamma_M )</td>
<td>( \gamma_M )</td>
<td>( \ldots )</td>
<td>( \gamma_M )</td>
</tr>
</tbody>
</table>
Another simple choice: piecewise-constant volatilities

\[
\forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_{i, \beta(t)} \text{ constant.} \tag{55}
\]

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1(t) )</td>
<td>([0, T_0] )</td>
<td>( \gamma_{1,1} )</td>
</tr>
<tr>
<td>( L_2(t) )</td>
<td>([T_0, T_1] )</td>
<td>( \gamma_{2,1} ), ( \gamma_{2,2} )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>([T_1, T_2] )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( L_M(t) )</td>
<td>([T_{M-2}, T_{M-1}] )</td>
<td>( \gamma_{M,1} ), ( \gamma_{M,2} ), ( \gamma_{M,3} )</td>
</tr>
</tbody>
</table>
Specifying Libor volatilities (III)

Simpler: piecewise-constant volatility that depends only on the time to maturity

\[ \forall i = 1, \ldots, M, \quad \gamma_i(t) = \gamma_{i, \beta(t)} = \eta_i - (\beta(t) - 1) \text{ constant.} \quad (56) \]

<table>
<thead>
<tr>
<th></th>
<th>[0, (T_0)]</th>
<th>([T_0, T_1])</th>
<th>([T_1, T_2])</th>
<th>(\ldots)</th>
<th>([T_{M-2}, T_{M-1}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_1(t))</td>
<td>(\eta_1)</td>
<td>dead</td>
<td>dead</td>
<td>(\ldots)</td>
<td>dead</td>
</tr>
<tr>
<td>(L_2(t))</td>
<td>(\eta_2)</td>
<td>(\eta_1)</td>
<td>dead</td>
<td>(\ldots)</td>
<td>dead</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(L_M(t))</td>
<td>(\eta_M)</td>
<td>(\eta_{M-1})</td>
<td>(\eta_{M-2})</td>
<td>(\ldots)</td>
<td>(\eta_1)</td>
</tr>
</tbody>
</table>
Specifying Libor volatilities (IV)

Parametric choices

One may also define Libor volatilities with $\forall i = 1, \ldots, M,$ : 

$$\gamma_i(t) = [a(T_{i-1} - t) + d] e^{-b(T_{i-1} - t)} + c \quad (57)$$

or 

$$\gamma_i(t) = \eta_i [a(T_{i-1} - t) + d] e^{-b(T_{i-1} - t)} + c \quad (58)$$
Dynamics of the forward Libor rates under a unique forward measure

**Proposition**

Let $i \in \{1, \ldots, M\}$. The dynamics of the forward Libor rates $L_i(t)$ under the forward measure $Q^{T_k}$, $k = 1, \ldots, M$ is given by the following SDE:

If $k < i$,

$$
\frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW_{i,Q^{T_k}}^i - \sum_{j=k}^{i} \rho_{ij} \gamma_i(t) \gamma_j(t) \frac{\theta_j L_j(t)}{1 + \theta_j L_j(t)} dt,
$$

If $k = i$,

$$
\frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW_{i,Q^{T_i}}^i
$$

If $k > i$,

$$
\frac{dL_i(t)}{L_i(t)} = \gamma_i(t) dW_{i,Q^{T_k}}^i + \sum_{j=i+1}^{k} \rho_{ij} \gamma_i(t) \gamma_j(t) \frac{\theta_j L_j(t)}{1 + \theta_j L_j(t)} dt.
$$
Introducing the spot Libor measure

**Definition**

The spot Libor numeraire is defined as:

\[
B(t) = \frac{P(t, T_{\beta(t)-1})}{\beta(t)-1} \prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)
\]  
(59)

**Proposition**

Under the spot Libor measure \( Q^B \) associated with the numeraire \( B(t) \), the dynamics of the forward Libor \( L_i(t) \) is written:

\[
\frac{dL_i(t)}{L_i(t)} = \gamma_i(t)dW^{Q^B} + \sum_{j=\beta(t)}^{i} \rho_{ij} \gamma_i(t) \gamma_j(t) \frac{\theta_j L_j(t)}{1 + \theta_j L_j(t)} dt.
\]  
(60)
Swap market model

Original paper

Model definition
A swap market models assumes that the swap rate $S_{\alpha,\beta}$ is solution of the SDE:

$$\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} = \gamma_{\alpha,\beta}(t)dW^{Q_{\alpha,\beta}}(t), \quad (61)$$

where $Q_{\alpha,\beta}$ is the measure linked with numeraire $\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$.

- Compatibility with the Black formula for swaption
- Theoretical inconsistency with the BGM market model
Other LIBOR approaches
